

# Stability under constantly acting perturbations for difference equations and averaging

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## 1. Introduction.

In 1967 Banfi established in [3] that uniform asymptotic stability of a solution of an averaged ordinary differential equation implies closeness of solutions of an exact and an averaged equations on an infinite interval provided the solutions have close initial conditions. Similar results have been obtained for integro-differential equations by Filatov in [8], for ordinary differential equations with slow and fast time by Sethna in [13], and for functional differential equations of retarded type by Burd in [5].

In this paper we consider the problem of closeness of solutions of an exact and an averaged difference equations on an infinite interval. Appropriate assertions are derived from one special theorem on the stability under constantly acting perturbations. We note that the stability under constantly acting perturbations is sometimes called the total stability (see, Agarwal [1], Hahn [9]).

## 2. Theorem on the stability under constantly acting perturbations.

**Basic notation.** We will use the following notation:  $|x|$  is a norm of vector  $x \in \mathcal{R}^m$ ,  $\mathcal{N}$  is the set of nonnegative integers,  $B_x(K) = \{x : x \in \mathcal{R}^m, |x| \leq K\}$ ,  $G = \mathcal{N} \times B_x(K)$ . Let  $f(n, x)$  be a function that is defined on  $G$  with values in  $\mathcal{R}^m$  and is bounded in the norm. Let  $N \in \mathcal{N}$ . Let us assume

$$S_{x,N}(f) = \sup_{n \in \mathcal{N}} \left| \sum_{k=n}^{n+N-1} f(k, x) \right|, \quad x \in B_x(K).$$

**Lemma.** *Let the function  $f(n, x)$  be continuous in  $x$  uniformly with respect to  $n \in \mathcal{N}$ . Assume that values of the function  $x(n)$ ,  $(n \in \mathcal{N})$  belong to  $B_x(K)$ .*

*Then for any  $\eta > 0$  there exists a number  $\varepsilon > 0$  such that*

$$\sup_{n \in \mathcal{N}} \left| \sum_{k=n}^{n+N-1} f(k, x(k)) \right| < \eta,$$

*if*

$$S_{x,N}(f) < \varepsilon$$

**Proof.** Based on the conditions of the Lemma there exists  $\delta > 0$  such that  $|f(n, x_1) - f(n, x_2)| < \eta$ , if  $|x_1 - x_2| < \delta$ . We denote by  $x^0(n)$  a function with values in  $B_x(K)$  that has a finite set of values and satisfies  $|x(n) - x^0(n)| < \delta$ ,  $n \in \mathcal{N}$ . Such function  $x^0(n)$

exists as  $B_x(K)$  is compact. The function  $x^0(n)$  has no more than  $l$  different values, where number  $l$  depends only on  $\delta$ . The statement of the Lemma follows from the inequality

$$\left| \sum_{k=n}^{n+N-1} f(k, x(k)) \right| \leq \left| \sum_{k=n}^{n+N-1} [f(k, x(k)) - f(k, x^0(k))] \right| + \left| \sum_{k=n}^{n+N-1} f(k, x^0(k)) \right|.$$

### 3. The main theorem.

We consider the following difference equation in  $\mathcal{R}^m$

$$\Delta x(n) = X(n, x) + R(n, x), \quad n = n_0, n_0 + 1, \dots, \quad (1)$$

where  $\Delta x(n) = x(n+1) - x(n)$  and functions  $X(n, x)$  and  $R(n, x)$  are defined on  $G$ .

Alongside with the equation (1) we consider the unperturbed difference equation

$$\Delta y(n) = X(n, y). \quad (2)$$

We suppose that equation (2) has a solution  $\psi(n, n_0, \xi_0)$  ( $\psi(n_0, n_0, \xi_0) = \xi_0$ ) which is defined for all  $n \geq n_0$  and  $\psi(n, n_0, \xi_0)$  together with its  $\rho$ -neighborhood ( $\rho > 0$ ) remains in the interior of the set  $G$ .

**Theorem 1.** *Let function  $X(n, x)$  be bounded on  $G$  and satisfies the Lipschitz condition*

$$|X(n, x_1) - X(n, x_2)| \leq L|x_1 - x_2|, \quad x_1, x_2 \in B_x(K). \quad (3)$$

*Let function  $R(n, x)$  be continuous in  $x$  uniformly with respect to  $n \in \mathcal{N}$  and be bounded on  $G$ . Let a solution  $\psi(n, n_0, \xi_0)$  of equation (2) be uniformly asymptotically stable.*

*Then for any  $\varepsilon > 0$  ( $0 < \varepsilon < \rho$ ) there exists  $N \in \mathcal{N}$  and numbers  $\eta_1(\varepsilon)$ ,  $\eta_2(\varepsilon)$  such that for all solutions  $x(n, n_0, x_0) \in B_x(K)$  ( $x(n_0, n_0, x_0) = x_0$ ) of equation (1) with initial values satisfying inequality*

$$|x_0 - \xi_0| < \eta_1(\varepsilon)$$

*and for all functions  $R(n, x)$  satisfying inequality*

$$S_{x,N}(R) < \eta_2(\varepsilon)$$

*the inequality*

$$|x(n, n_0, x_0) - \psi(n, n_0, \xi_0)| < \varepsilon, \quad n \geq n_0 \quad (4)$$

*holds.*

**Proof.** Let  $y(n, n_0, x_0)$  be a solution of equation (2) with the same initial condition as the solution  $x(n, n_0, x_0)$  of equation (1). These solutions satisfy the following equations respectively

$$y(n, n_0, x_0) = x_0 + \sum_{k=n_0}^{n-1} X(k, y(k, n_0, x_0)),$$

$$x(n, n_0, x_0) = x_0 + \sum_{k=n_0}^{n-1} [X(k, x(k, n_0, x_0)) + R(k, x(k, n_0, x_0))].$$

It follows the inequality

$$|x(n, n_0, x_0) - y(n, n_0, x_0)| \leq \sum_{k=n_0}^{n-1} |X(k, x(k, n_0, x_0)) - X(k, y(k, n_0, x_0))| + \left| \sum_{k=n_0}^{n-1} R(k, x(k, n_0, x_0)) \right|.$$

Using condition (3) of Theorem we obtain

$$|x(n, n_0, x_0) - y(n, n_0, x_0)| \leq L \sum_{k=n_0}^{n-1} |x(k, n_0, x_0) - y(k, n_0, x_0)| + f(n),$$

where

$$f(n) = \left| \sum_{k=n_0}^{n-1} R(k, x(k, n_0, x_0)) \right| = \left| \sum_{k=n_0}^{n_0+N-1} R(k, x(k, n_0, x_0)) \right|,$$

and  $N = n - n_0$ . A well known inequality (see, for example, [2]) implies

$$|x(n, n_0, x_0) - y(n, n_0, x_0)| \leq f(n) + L \sum_{k=n_0}^{n-1} f(k)(1 + L)^{n-1-k}$$

Therefore for  $n_0 \leq n \leq n_0 + N$  the upper bound for

$$|x(n, n_0, x_0) - y(n, n_0, x_0)|$$

depends on the values  $f(n)$  ( $n = n_0, \dots, n_0 + N - 1$ ). From uniform asymptotic stability of solution  $\psi(n, n_0, \xi_0)$  of equation (2) follows that there exist numbers  $\delta < \varepsilon$  and  $T \in \mathcal{N}$  such that inequality  $|x_0 - \xi_0| < \delta$  implies

$$\begin{aligned} |y(n, n_0, x_0) - \psi(n, n_0, \xi_0)| &< \frac{\varepsilon}{2} \quad n \geq n_0, \\ |y(n_0 + T, n_0, x_0) - \psi(n_0 + T, n_0, \xi_0)| &< \frac{\delta}{2}. \end{aligned} \tag{5}$$

We now set  $N = T$ . The Lemma implies that we can find a number  $\eta_2(\varepsilon)$  such that

$$|x(n, n_0, x_0) - y(n, n_0, x_0)| < \frac{\delta}{2}, \quad n_0 \leq n \leq n_0 + T. \tag{6}$$

holds. Then

$$|x(n, n_0, x_0) - \psi(n, n_0, \xi_0)| < \frac{\varepsilon}{2} + \frac{\delta}{2} < \varepsilon, \quad n_0 \leq n \leq n_0 + T.$$

Furthermore, (5) and (6) imply

$$|x(n_0 + T, n_0, x_0) - \psi(n_0 + T, n_0, \xi_0)| < \delta.$$

Hence for the interval  $[n_0, n_0 + T]$  the solution  $x(n, n_0, x_0)$  remains in  $\varepsilon$ -neighborhood of the solution  $\psi(n, n_0, \xi_0)$  and at the moment  $n = n_0 + T$  belongs to the  $\delta$ -neighborhood of  $\psi(n, n_0, \xi_0)$ .

We now consider  $n = n_0 + T$  as an initial moment. Using the same arguments as above we obtain

$$|x(n, n_0, x_0) - \psi(n, n_0, \xi_0)| \leq \varepsilon, \quad n_0 + T \leq n \leq n_0 + 2T$$

and

$$|x(n_0 + 2T, n_0, x_0) - \psi(n_0 + 2T, n_0, \xi_0)| \leq \delta.$$

Repetitive application of the same argument completes the proof of the Theorem.

We note that last part of proof of the Theorem 1 uses the reasoning similar to the Lemma 6.3 from [4].

**Remark 1.** The statement of Theorem 1 differs from the statements of known theorems on the stability under constantly acting perturbations [1, 9 - 12] in using a more general assumption on the perturbation  $R(n, x)$ . Theorem 1 implies Halanay's theorem, if number  $\eta_2(\varepsilon) = N\delta_2(\varepsilon)$ , where  $\delta_2(\varepsilon)$  is a number from definition 5.13.1 [1]. If we assume

$$S_{x,N}(f) = \sup_{n \in \mathcal{N}} \sum_{k=n}^{n+N-1} |f(k, x)|, \quad x \in B_x(K),$$

then we obtain a difference analog of the Theorem 24.1 from [11].

**Remark 2.** We start with the following definition.

**Definition.** The solution  $\psi(n, n_0, \xi_0)$  is called uniformly asymptotically stable with respect to a part of the variables  $\psi_1, \dots, \psi_k$ ,  $k < m$ , if its asymptotically stable in the sense of Lyapunov with respect to a part of the variables  $\psi_1, \dots, \psi_k$ ,  $k < m$  and if for any number  $\gamma > 0$  there exists a number  $T(\gamma) \in \mathcal{N}$ , such that for the solution  $y(n, n_0, x_0)$  of the equation (2) is satisfied inequality

$$|y_i(n, n_0, x_0) - \psi_i(n, n_0, \xi_0)| < \gamma, \quad n \geq n_0 + T(\gamma), \quad i = 1, \dots, k,$$

for any initial moment  $n_0$  and any initial values  $x_0$  from the domain of the attraction of solution  $\psi(n, n_0, \xi_0)$  with respect to a part of the variables (i.e. from domain where is satisfied the limit equality

$$\lim_{n \rightarrow \infty} |y_i(n, n_0, x_0) - \psi_i(n, n_0, \xi_0)| = 0, \quad i = 1, \dots, k.)$$

If  $k = m$ , the definition above coincides with the definition of uniform asymptotic stability (see, for example, [6]). A detailed discussion of the stability theory with respect to a part of the variables is given, for example, in [14].

If the solution  $\psi(n, n_0, \xi_0)$  of equation (2) uniformly asymptotically stable only with respect to a part of the variables  $\psi_1, \dots, \psi_k$ ,  $k < m$ , then inequality (4) in the statement of the Theorem 1 can be replaced with the inequality

$$|x_i(n, n_0, x_0) - \psi_i(n, n_0, \xi_0)| < \varepsilon, \quad i = 1, \dots, k.$$

#### 4. Averaging on an infinite interval.

Theorem 1 is applicable to the problem of averaging on an infinite interval for difference equations.

We consider the following difference equation in  $\mathcal{R}^m$

$$\Delta x(n) = \varepsilon X(n, x), \tag{7}$$

where  $\varepsilon > 0$  is a small parameter,  $X(n, x)$  is defined for  $(n, x) \in G$ .

**Theorem 2.** *Let*

- 1) *function  $X(n, x)$  be continuous in  $x$  uniformly with respect to  $n \in \mathcal{N}$ ;*
- 2)  *$|X(n, x)| \leq M_1 < \infty$ ,  $(n, x) \in G$ ;*
- 3) *the limit*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=n}^{n+N-1} X(k, x) = \bar{X}(x)$$

*exists uniformly with respect to  $n$  for any  $(n, x) \in G$  and  $\bar{X}(x)$  be bounded in the norm*

$$|\bar{X}(x)| \leq M_2 < \infty, \quad x \in B_x(K);$$

- 4) *function  $\bar{X}(x)$  satisfies the Lipschitz condition*

$$|\bar{X}(x_1) - \bar{X}(x_2)| \leq L|x_1 - x_2|, \quad x_1, x_2 \in B_x(K),$$

- 5) *the averaged equation*

$$\Delta x(n) = \varepsilon \bar{X}(x)$$

*has a uniformly asymptotically stable solution  $\psi(n, n_0, \xi_0)$  (uniformly asymptotically stable with respect to a part of the variables  $x_1, \dots, x_k$ ,  $k < m$ ), which with its  $\rho$ -neighborhood ( $\rho > 0$ ) belong to  $G$ .*

*Then for any  $\alpha$  ( $0 < \alpha < \rho$ ) there exists  $\varepsilon_1(\alpha)$  ( $0 < \varepsilon_1 < \varepsilon_0$ ) and  $\beta(\alpha)$  such that for all  $0 < \varepsilon < \varepsilon_1$  the solution  $\varphi(n, n_0, x_0) \in B_x(K)$  of equation (7) with an initial condition satisfying inequality*

$$|x_0 - \xi_0| < \beta(\alpha) \quad (|x_{0i} - \xi_{0i}| < \beta(\alpha), i = 1, \dots, k < m)$$

*we have*

$$|\psi(n, n_0, \xi_0) - \varphi(n, n_0, x_0)| < \alpha, \quad n \geq n_0$$

$$(|\psi_i(n, n_0, \xi_0) - \varphi_i(n, n_0, x_0)| < \alpha, \quad i = 1, \dots, k < m \quad n \geq n_0).$$

The equation (7) can be written in the form

$$\Delta x(n) = \varepsilon \bar{X}(x) + \varepsilon R(n, x),$$

where  $R(n, x) = X(n, x) - \bar{X}(x)$ . We show that Theorem 2 follows from Theorem 1. Given  $\alpha$  we choose  $N(\alpha) = [\frac{1}{\varepsilon}]$  where  $[x]$  is the integer part of  $x$ . Then

$$\left| \varepsilon \sum_{k=n}^{n+[\frac{1}{\varepsilon}]-1} R(k, x) \right| \leq \frac{1}{[\frac{1}{\varepsilon} - 1]} \left| \sum_{k=n}^{n+[\frac{1}{\varepsilon-1}]} R(k, x) \right|.$$

Therefore condition 3) of the Theorem 2 implies that for a sufficiently small  $\varepsilon$  the function  $R(n, x)$  satisfies of conditions of Theorem 1.

**4. Averaging on an infinite interval of systems with the right-hand side that vanishes over time.**

We now consider the following difference equation in  $\mathcal{R}^m$

$$\Delta x(n) = \frac{1}{n} X(n, x), \quad n = n_0, n_0 + 1, \dots, \quad (8)$$

where  $X(n, x)$  is defined for  $(n, x) \in G$ .

**Theorem 3.** *Let*

- 1) *function  $X(n, x)$  be continuous in  $x$  uniformly with respect to  $n \in \mathcal{N}$ ;*
- 2)  *$|X(n, x)| \leq M_1 < \infty, \quad (n, x) \in G$ ;*
- 3) *the limit*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=n}^{n+N-1} X(k, x) = \bar{X}(x)$$

*exist uniformly with respect to  $n$  for any  $(n, x) \in G$  and  $\bar{X}(x)$  be bounded in the norm*

$$|\bar{X}(x)| \leq M_2 < \infty, \quad x \in B_x(K);$$

- 4) *function  $\bar{X}(x)$  satisfies the Lipschitz condition*

$$|\bar{X}(x_1) - \bar{X}(x_2)| \leq L|x_1 - x_2|, \quad x_1, x_2 \in B_x(K),$$

- 5) *the averaged equation*

$$\Delta x(n) = \frac{1}{n} \bar{X}(x)$$

has a uniformly asymptotically stable solution  $\psi(n, n_0, \xi_0)$  (uniformly asymptotically stable with respect to a part of variables  $x_1, \dots, x_k$ ,  $k < m$ ), which with its  $\rho$ -neighborhood ( $\rho > 0$ ) is contained in  $G$ .

Then for any  $\alpha$  ( $0 < \alpha < \rho$ ) there exists  $\varepsilon_1(\alpha)$  ( $0 < \varepsilon_1 < \varepsilon_0$ ) and  $\beta(\alpha)$  such that for all  $0 < \varepsilon < \varepsilon_1$  the solution  $\varphi(n, n_0, x_0) \in B_x(K)$  of equation (8), with initial condition satisfying inequality

$$|x_0 - \xi_0| < \beta(\alpha) \quad (|x_{0i} - \xi_{0i}| < \beta(\alpha), i = 1, \dots, k)$$

we have

$$\begin{aligned} |\psi(n, n_0, \xi_0) - \varphi(n, n_0, x_0)| &< \alpha, \quad n \geq n_0 \\ (|\psi_i(n, n_0, \xi_0) - \varphi_i(n, n_0, x_0)| &< \alpha, \quad i = 1, \dots, k < m \quad n \geq n_0). \end{aligned}$$

The proof of the Theorem 3 is quite similar to the proof of the Theorem 2.

### 5. Dynamics of selection of genetic population in a varying environment.

As an example we consider dynamics of a selection of a Mendelian population with a genetic pool made of only two alleles, that we'll call  $A$  and  $a$ . We assume that the fitness of the genotypes  $AA$ ,  $Aa$ ,  $aa$  are  $1 - \varepsilon\alpha(n)$ ,  $1$ ,  $1 - \varepsilon\beta(n)$  respectively. Here  $n$  is number of the generation,  $\varepsilon > 0$  is a small parameter,  $\alpha(n)$ ,  $\beta(n)$  are periodic functions with period  $l \in \mathcal{N}$  and positive mean values. Let  $p_n$ ,  $q_n$  be the frequencies of alleles  $A$ ,  $a$  in generation  $n$  respectively. The evolution equation has the form (see, for example, [7])

$$\Delta p_n = \varepsilon p_n(1 - p_n) \frac{\beta(n) - (\alpha(n) + \beta(n))p_n}{1 - \varepsilon[(\alpha(n) + \beta(n))p_n^2 + 2\beta(n)p_n - \beta(n)]}. \quad (9)$$

The averaged equation

$$\Delta \bar{p}_n = \varepsilon \bar{p}_n(1 - \bar{p}_n)(\beta_0 - (\alpha_0 + \beta_0)\bar{p}_n)$$

has a unique asymptotically stable equilibrium

$$\bar{p} = \frac{\beta_0}{\alpha_0 + \beta_0},$$

where  $\alpha_0$ ,  $\beta_0$  are mean value of periodic functions  $\alpha(n)$ ,  $\beta(n)$  accordingly.

Then for a sufficiently small  $\varepsilon$  equation (9) has an asymptotically stable periodic solution with period  $l \in \mathcal{N}$  (see [10]). Theorem 2 implies that for any  $\delta > 0$  there exists  $\eta(\delta)$  such that the solution  $p_n(0, x_0)$  of equation (9), with initial condition satisfying inequality

$$|x_0 - \xi_0| < \eta(\delta),$$

where  $\xi_0 > 0$ ,  $\xi_0 \neq \frac{\beta_0}{\alpha_0 + \beta_0}$  the inequality

$$|p_n(0, x_0) - \bar{p}_n(0, \xi_0)| < \delta, \quad n \geq 0$$

holds.

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